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The spherical model for spin glasses revisited

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Abstract. We consider the spherical model for a one-component spin glass in the mean-field limit. Using the replica method we can solve the model exactly, allowing explicitly for replica symmetry breaking. The solution is replica symmetric and marginally stable. However, perturbations around the spherical model, in particular the introduction of an on-site Ising-like spin length probability, destabilise the replica symmetric solution and give rise to a new solution. The mass of the fluctuations destroying replica symmetry is shown to depend on the fourth cumulant of the spin length distribution for a given site.

1. Introduction

The infinite-range spherical model for spin glasses, like the corresponding Ising model [1], possesses a saddlepoint solution which is *exact* in the thermodynamic limit. The model, furthermore, can be solved without replicas, and was the toy model introduced by Kosterlitz *et al* [2] to show that the replica trick could indeed reproduce the correct free energy and critical behaviour. We therefore consider the spherical model as a good starting point from which to treat the problem of replica symmetry breaking, and the origins of the destabilisation of the replica symmetric solution. We begin by formulating the model in the most general terms possible, within the context of the replica method, and show that this results in the replica symmetric solution. To understand the effect of perturbations, however, the model is specialised to looking at a particular class of solutions given by the the Parisi parametrisation [3] which is introduced here once again, as in the well known SK model, essentially by fiat. Doing this enables us to take the replica limit in a well regulated way, allowing for a generalisation of the original solution, and making it possible to understand the effect of perturbations around the starting model. The completion of the programme in terms of a full analysis that does not involve this parametrisation remains for the future. Here we will find, upon making this ansatz, that the passage from the spherical to the Ising limit by way of limiting on-site spin fluctuations results in destabilisation of the original solution. The replicon modes are identified and their masses are seen to be proportional to the fourth cumulant of the spin length distribution.

Finally, we re-express the effective theory in terms of a new function, which greatly facilitates the calculation of the equation of state. This function can be related to the probability distribution $P(Q)$ that was introduced by Parisi [4] in a replica-free description of the statistical mechanics of spin glasses.

Sections 2–4 present the model and its solution. The fifth section discusses the passage to the Ising limit. In the sixth section, an alternative formulation is presented for the saddlepoint calculation.

2. The spherical model

The spherical model for N one-component spins $\{S_i, i = 1, N\}$ is given by the Hamiltonian

$$H = \frac{1}{2}\sum_{i \leq j} J_{ij} S_i S_j \tag{1}$$

with the spin lengths satisfying the constraint

$$\sum_{i=1}^{N} S_i^2 = N \tag{2}$$

and the exchange couplings $\{J_{ij}\}$ are taken from a Gaussian distribution. For all pairs of spins in the system

$$P[J_{ij}] = \left(\frac{2\pi J^2}{N}\right)^{-1/2} \exp\left(\frac{-N J_{ij}^2}{2J^2}\right). \tag{3}$$

If the system is self-averaging, properties such as the equation of state can be computed from the average of its free energy over all realisations of the random couplings $\{J_{ij}\}$. This is done by the replica trick [5] which relies on the use of the identity $-\beta F_J = \ln Z_J = \lim_{n\to 0}[Z_J^n - 1]/n$. The partition function for a given set of $\{J_{ij}\}$ is $Z_J = \mathrm{Tr}\exp(-\beta \sum J_{ij} S_i S_j)$. Introducing replicas, and performing an average over the random couplings results in an effective four spin interaction that couples spins in different replicas. This term is decoupled using the standard Hubbard–Stratonovich transformation along with the introduction of the fields $\{Q_{\alpha\beta}\}$, leading to an effective theory after random averaging given by

$$Z_{\mathrm{eff}} = \exp\left(\frac{nN\beta^2 J^2}{4}\right) \int \mathrm{D}Q_{\alpha\beta} \exp\left(-\frac{N}{4}\sum_{\alpha,\beta} Q_{\alpha\beta}^2\right) \mathrm{Tr}\exp\left(\frac{\beta J}{2}\sum_{\alpha,\beta}{}' Q_{\alpha\beta} S_{i\alpha} S_{i\beta}\right) \tag{4}$$

(terms with $\alpha = \beta$ are to be excluded from the primed sum).

In the thermodynamic limit, taking the number of sites N to infinity, the saddlepoint solution for $Q_{\alpha\beta}$ is exact and relates $Q_{\alpha\beta}$ to the spin overlap between replicas, $Q_{\alpha\beta} = \beta J/N \sum \langle S_{i\alpha} S_{i\beta}\rangle$ $(\alpha \neq \beta)$.

Using an integral representation of the spherical constraint in (2), it can be seen that the spin traces on the different sites can be decoupled. The resulting Gaussian integrals over the vectors $\{S_\alpha, \alpha = 1, n\}$ thus give

$$Z_{\mathrm{eff}} = \int \mathrm{D}Q_{\alpha\beta} \int \mathrm{D}z_\alpha \exp(-L_{\mathrm{eff}}[Q, z]) \tag{5}$$

where

$$-L_{\mathrm{eff}} = \frac{Nn\beta^2 J^2}{4} + \frac{Nn\ln 2\pi}{2} - \frac{N}{4}\sum_{\alpha,\beta} Q_{\alpha\beta}^2 + N\sum_{\alpha} z_\alpha - \frac{N}{2}\mathrm{Tr}\ln(2z_\alpha\delta_{\alpha\beta} - \beta J Q_{\alpha\beta}). \tag{6}$$

The $\{z_\alpha\}$ are the Lagrange multiplier fields corresponding to n constraints over the replicas. The trace in (6) refers to the space of replicas. Equation (6) is the effective theory for the spherical model that we now wish to study.

3. The replica symmetric solution

As a preliminary to the more general treatment that we will later introduce, we present the replica symmetric solution for the mean-field spherical model introduced in the previous section. This was done originally by Kosterlitz *et al* [2], who solved the model with and without the introduction of the replica trick, and showed that the free energy computed by both methods was the same, thereby validating the use of the replica technique in this particular instance. The replica symmetric solution proceeds as follows.

The assumption of replica symmetry allows us to solve for a saddlepoint at which $Q_{\alpha\beta} = Q = \beta J q \quad \forall(\alpha, \beta)$ and $z_\alpha = z$ independent of the replica indices. Substituting these into (5) and taking the limit $n \to 0$ yields the averaged free energy per spin (as $N \to \infty$)

$$-\beta f = \frac{Q^2}{4} + z - \frac{\ln(2z + \beta J Q)}{2} + \frac{\ln 2\pi}{2} + \frac{1}{2}\frac{\beta J Q}{2z + \beta J Q} + \frac{\beta^2 J^2}{4} \tag{7}$$

where the saddlepoint solutions for Q and z (valid for $N \to \infty$) are

$$\begin{aligned} &Q = \beta J - 1 \qquad z = \beta J(1 - \beta J/2) \qquad T < T_c \\ &Q = 0 \qquad z = \tfrac{1}{2} \qquad T > T_c \end{aligned} \tag{8}$$

($T_c = J$). Using the saddlepoint values in the expression for the free energy we find

$$\begin{aligned} -\beta f &= \beta J - \tfrac{1}{2}\ln \beta J - \tfrac{1}{2}(2 - \ln 2\pi) \qquad T < T_c \\ -\beta f &= \tfrac{1}{2}(1 + \ln 2\pi) + (\beta J/2)^2 \qquad T > T_c \end{aligned} \tag{9}$$

We now discuss the model in more general terms, keeping this preliminary solution in mind in what follows.

4. Solution in terms of eigenvalues; the Parisi parametrisation

Returning to the full partition function in (5), it is evident that the effective action L_{eff} can be written entirely in terms of invariants of $Q_{\alpha\beta}$ under rotations in replica space. This follows from the expansion of the Tr ln term in powers of Q. We can therefore express the path integral over $Q_{\alpha\beta}$ alternatively as an integral over the set of eigenvalues of Q-matrices (which we know to exist, the $Q_{\alpha\beta}$ being real symmetric matrices by construction). The elements of $Q_{\alpha\beta}$, being conjugate to the spin overlaps, are Gaussian distributed random variables (by the central limit theorem , as $N \to \infty$). In terms of the eigenvalues

$$\frac{-L_{\text{eff}}}{N} = \frac{n\beta^2 J^2}{4} + nz - \frac{1}{4}\sum \lambda_\alpha^2 - \frac{1}{2}\sum \ln(2z - \beta J \lambda_\alpha) + \frac{\ln 2\pi}{2} + \frac{1}{N}\ln \|J\|. \tag{10}$$

The last term arises from the Jacobian of the transformation. For the case of an orthogonal ensemble, this is given by $\|J\| = \Pi_{\alpha<\beta}|\lambda_\alpha - \lambda_\beta|$ [7] . Here, by virtue of the factor $1/N$ appearing in front, we shall drop this term in our calculation of the saddlepoint (recall n is finite at this stage). After extremisation, the limit $n \to 0$ is to be taken. In order to do this we find it necessary to introduce a specific parametrisation of the matrices $Q_{\alpha\beta}$ that will allow for the taking of this limit in a well defined manner. The saddlepoint calculation of the previous section was evidently a very restricted one because of the assumption of replica symmetry. While the solution thus obtained appears to be the correct one for the spherical model, we are motivated by the knowledge that it is not correct in the case of the Ising spin glass to try to enlarge the space of allowed solutions. We would like, in an ideal world, to solve the problem without making any assumptions as to the dependence on the replica index. However this is difficult to do in complete generality, and so we have resorted here to looking for solutions that satisfy a rather special property—that of ultrametricity [6]. Although the spherical model might not itself possess an ultrametric saddlepoint solution, the simple solution of the last section will be shown to be marginally stable with respect to fluctuations of the generalised order parameter $Q(x)$ and hence possibly vulnerable to perturbations.

The replica symmetry breaking scheme devised by Parisi provides one way of parametrising an $n \times n$ real symmetric matrix in the limit that n goes to zero. The scheme has been described in the litérature (see for example [8]), and involves a heirarchical construction of the matrix elements of Q described in the $n \to 0$ limit by a function of a single variable $Q(x)$, with $x \in [0, 1]$. The eigenvalues of the symmetric matrix thus parametrised can be written down in terms of the function $Q(x)$. The trace of arbitrary powers of Q, $\mathrm{Tr}\, Q^k = \sum \lambda_\alpha^k$ is given by the general formula

$$(-1)^k \sum \lambda_\alpha^k = \left(\int_n^1 \mathrm{d}x\, Q(x) \right)^k - n \int_n^1 \mathrm{d}x\, \frac{1}{x^2} \left(\int_x^1 \mathrm{d}s\, Q(s) + x\, Q(x) \right)^k \tag{11}$$

to lowest order in n. This allows us to express the effective action entirely in terms of $Q(x)$, and the result for the free energy is

$$\begin{aligned} -\beta f = \mathrm{ext}\Bigg\{ & \frac{\beta^2 J^2}{4} - \frac{\ln z/\pi}{2} + z - \frac{1}{4}\left[\left(\frac{\beta J}{2z} \right)^2 - 1 \right] \int_0^1 \mathrm{d}x\, Q^2(x) \\ & + \frac{1}{2} \sum_{k=3}^{\infty} \frac{1}{k} \left(-\frac{\beta J}{2z} \right)^k \Bigg[Q^k(1) - kQ(0) \left(\int_0^1 Q \right)^{k-1} \\ & - k \int_0^1 Q'(x) \left(\int_x^1 \mathrm{d}s\, Q(s) + x\, Q(x) \right)^{k-1} \Bigg] \Bigg\}. \end{aligned} \tag{12}$$

This can be simplified, by partial integration and summing terms, to the form

$$\begin{aligned} -\beta f = \mathrm{ext}\Bigg\{ & \frac{\beta^2 J^2}{4} + z + \frac{\ln 2\pi}{2} - \frac{\ln(2z + \beta J Q(1))}{2} + \frac{1}{4} \int_0^1 \mathrm{d}x\, Q^2(x) - \frac{\beta J Q(0)}{2[2z + \beta J \int_0^1 Q(x)]} \\ & - \frac{1}{2} \int_0^1 \mathrm{d}x\, Q'(x)\, \frac{\beta J}{2z + \beta J [\int_x^1 Q + xQ]} \Bigg\} + \mathrm{O}(1/N). \end{aligned} \tag{13}$$

The expression inside the braces can now be extremised by differentiation with respect to $Q(x)$ and z. After repeated differentiations, the extremal solution for $Q(x)$ is found to satisfy

$$Q'(x) = 0 \tag{14}$$

i.e., $Q(x)$ = constant is the only possible solution. One recovers the results for the saddlepoint of the previous section. The solution as formulated here corresponds to a family of stable solutions $Q_{\alpha\beta}$ that transform into each other under rotations in the space of replicas. We note that the physically relevant solution must be off-diagonal and symmetric due to the saddlepoint relationship between $Q_{\alpha\beta}$ and the spin overlaps.

5. Replica symmetry breaking perturbations. Stability under Gaussian fluctuations and approach to the Ising model

We now address the question of how replica symmetry might be broken in this and related models for the spin glass. Going back to the effective theory in terms of Q and z given by (12), it is straightforward to expand in powers of Q, which appears to all orders in L_{eff}. We may now consider the effect of adding small perturbations to the effective action, in order to investigate the stability of the replica symmetric solution, and the origin of replica symmetry breaking in the related Ising spin glass.

Consider the free energy of (12) close to T_c. Keeping the first few terms in the expansion in powers of $Q(x)$ we will show the corresponding extremal solution for this function to be replica symmetric . This is true, of course, to all orders in $Q(x)$, being the exact solution found in the last section. To this truncated free energy now, however, we may consider the effect of adding perturbing terms, with the intent of studying the fate of the replica symmetric solution. In particular, we wish to study the relation of the this model to the infinite-range Ising model, which has been shown to admit a non-trivial solution for $Q(x)$ [3]. We will show here that while the addition of any perturbing terms up to third order in the function $Q(x)$ do not give rise to replica symmetry breaking, we can find perturbations of order Q^4 that do give rise to a new replica symmetry broken solution in addition to the constant solution. Which of these is actually chosen must then be determined by a stability analysis with respect to general fluctuations in the space of functions $Q(x)$. This is of interest because such quartic terms would in fact be generated if we were to start out with a different length distribution for the spins than that given by the spherical constraint.

The truncated free energy up to fourth order in $Q(x)$ for the pure spherical model is

$$\begin{aligned}-\beta f = \frac{\beta^2 J^2}{4} - \frac{\ln z/\pi}{2} + z + \frac{1}{4}\left[1 - \left(\frac{\beta J}{2z}\right)^2\right] \int_0^1 Q^2(x) \\ - \frac{1}{6}\left(\frac{\beta J}{2z}\right)^3 \left[Q^3(1) - 3Q(0)\left(\int_0^1 Q\right)^2 - 3\int \mathrm{d}x\, Q'\left(\int_x^1 Q + xQ\right)^2\right] \\ + \frac{1}{8}\left(\frac{\beta J}{2z}\right)^4 \left[Q^4(1) - 4Q(0)\left(\int_0^1 Q\right)^3 - 4\int \mathrm{d}x\, Q'\left(\int_x^1 Q + xQ\right)^3\right]. \quad (15)\end{aligned}$$

Differentiating this with respect to $Q(x)$, as we already said in the last section, results in the only possible solution for $Q(x)$ being the replica symmetric one $Q = \tau + \tau^2$, which is the solution in (8) expanded close to T_c ($\tau = (T_c - T)/T_c$.) It can be checked that the addition of terms of linear, quadratic or cubic order results in a solution that satisfies $Q'(x)$=0. The quartic perturbations do result in an additional type of extremising solution. There are several types of terms one can have at this order. One relatively innocuous term is $\mathrm{Tr}Q^4$, which does not result in a new solution. Other terms such as $\sum_{\alpha\neq\gamma} Q^2_{\alpha\beta}Q^2_{\alpha\gamma}$ and $\sum Q^4_{\alpha\beta}$ can be seen to result in an additional saddlepoint solution. If we add the term $u\sum Q^4_{\alpha\beta}$ to (15), the extremal solution satisfies the equations: $Q'(x) = 0$ or

$$(8u + x^2)Q''(x) + 3xQ'(x) = 0. \tag{16}$$

The latter has the solution $Q(x) = c_1 x/\sqrt{x^2+8u} + c_2$ where the constants can be determined by the set of saddlepoint equations, as well as the region in the interval in which $Q(x)$ is non-constant. The solution is

$$Q(x) = \begin{cases} (1/6\sqrt{2u})(x/\sqrt{x^2+8u}) & 0 \le x < x_1 \\ Q(x_1) & x_1 \le x \le 1 \end{cases} \tag{17}$$

where $x_1 = 24u\tau$. Comparison with the solution of the SK model of Thouless *et al* [9] shows them to be very similar. It is not yet established whether this is a stable solution. To do this we must take the functional derivative of (15) twice. In the absence of the quartic perturbation uQ^4, the second derivative is

$$\delta^2(-\beta f) = 2(\tau+\tau^2)\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, \delta Q(x)\delta Q(y) = (\tau+\tau^2)\left(\int_0^1 \mathrm{d}x\, \delta Q(x)\right)^2 \tag{18}$$

where we have used the replica symmetric saddlepoint solution to simplify the second derivative (see the appendix). This says that the solution is (marginally) stable, the eigenvalues of the fluctuations-matrix being positive. When the quartic term is present, however, the second derivative can be shown to result in (keeping terms up to τ^2)

$$\delta^2(-\beta f) = -12u\tau^2 \int_0^1 [\delta Q(x)]^2 + \frac{(\tau+\tau^2)}{2}\left(\int_0^1 \delta Q(x)\right)^2. \tag{19}$$

Clearly there exist fluctuations such that the first term, although of higher order in τ, is greater than the second term. If u is positive, the fluctuations will destroy the replica symmetric solution.

As remarked already, the way we choose to modify the model in order to generate additional terms in the expansion (15) is in the choice of the spin length distribution. For the spherical model, (4) can be rewritten

$$Z_{\mathrm{eff}} = \exp\left(\frac{nN\beta^2J^2}{4} + \frac{Nn\ln 2\pi}{2}\right)\int \mathrm{D}Q_{\alpha\beta}\int \mathrm{D}z \exp\left(zN - \frac{N}{4}\sum Q^2_{\alpha\beta}\right)$$
$$\times \left[\int \prod_\alpha \mathrm{d}S_\alpha\, P(S_\alpha) \exp\left(\beta J \sum Q_{\alpha\beta}S_\alpha S_\beta\right)\right]^N. \tag{20}$$

The spin length probability distribution here is $P[S] = \sqrt{z/\pi}\,\exp(-zS^2)$. The spin weight is Gaussian, and is maximum around a spin length of zero, although the spherical constraint ensures that the spins cannot all take on zero values. Spins at the more frustrated sites can, however, minimise energy by taking on extremely small values. The frustration can be made more important if we now force the spins to prefer non-zero values. More specifically, we now multiply the Gaussian weight above by a factor that favours $S = \pm 1$ (choosing the value 1 for convenience) :

$$P[S] = C \exp(-zS^2) \exp[-a(S^2 - 1)^2] \tag{21}$$

where a is a small parameter and C is the constant of normalisation. This modified spin distribution resembles the spherical model for small a ($a \ll z^2$) while for very large values of a it approaches the Ising limit. The object here is to study the perturbations induced by very small values of a around the spherical model. Using (21) to evaluate the spin trace in (20), we can find the prefactors of the powers of $Q_{\alpha\beta}$ appearing in the linked cluster expansion. The single-loop diagrams result in terms of the form Tr Q^k, which are those appearing in the unperturbed spherical model. In addition now, there is a term $\sum Q^4_{\alpha\beta}$ whose prefactor is proportional to the square of the fourth cumulant [3]. This prefactor, which vanishes for the Gaussian distribution, can be evaluated for (21) and gives an additional term in the free energy

$$H_{\text{pert}} = \frac{3a^2}{2z} \sum Q^4_{\alpha\beta} \tag{22}$$

to lowest order in the expansion in a. It is seen that the coefficient (called u in the discussion above) is positive definite. For any non-zero value of the parameter a, therefore, the replica symmetric solution is always destabilised according to (19).

In the case of the Ising model, we recall a similar situation existing in the field theory of the spin glass [10, 11], where the Ginzburg–Landau expansion contains a $Q_{\alpha\beta}$ term. It was noted that the origin of the destabilisation of the replica symmetric mean-field solution could be traced to the fact that this term occurs with a negative prefactor. A similar mechanism for replica symmetry breaking was shown to appear in the short-ranged random anisotropy model (RAM), where the infinite-N (here N stands for the number of components of the spin) limit had a replica symmetric solution, but was shown to be unstable at finite values of N due to the appearance of a destabilising quartic term at order $1/N$ [12]. Thus the result we have obtained in the case of the constrained spherical model, equation (19), is in keeping with this generally observed (but not understood) fact. Here we have a means of tuning the frustration, so to speak, and therefore of changing the strength of the replica symmetry breaking instability.

Equations (18) and (19) give the second variation of the free energy for the unmodified spherical model and its modified version, respectively. Writing this quantity as a sum over the eigenvalues of the eigenmodes of fluctuations around the saddlepoint, one has the result that the eigenvalue spectrum for the spherical model is

$$\lambda_1 = \tfrac{1}{2}\tau + \tfrac{1}{2}\tau^2 \tag{23}$$

corresponding to the uniform mode ($\delta Q(x) =$ constant), and furthermore

$$\lambda = 0 \tag{24}$$

for all the modes that satisfy $\int Q(x)\,\mathrm{d}x = 0$. These latter marginal modes are the ones which destabilise the replica symmetric solution when the spin length distribution is modified as described above.

6. Saddlepoint in terms of $A(Q)$

Here an alternative approach is presented for the analysis of perturbations. A new and useful function $A(q)$ can be defined, and it turns out to greatly facilitate the solution of the equations of state for the 'perturbed' spherical model spin glass. This function can also be given a meaning in the context of Parisi's explication of the generalised spin-glass order parameter, $Q(x)$. It is defined as follows:

$$F(x) = xQ(x) + \int_x^1 \mathrm{d}x' Q(x') \qquad A(Q) = F(x(Q)) \tag{25}$$

where $x(Q)$ can be found by inverting $Q(x)$. Conversely, given the function $A(Q)$ we can use (25) to find $x(Q)$ from

$$x(Q) = \frac{\mathrm{d}A(Q)}{\mathrm{d}Q} \tag{26}$$

so the function $A(Q)$ contains sufficient information to obtain the full x-dependence of $Q(x)$. It will be recalled that an important step in the interpretation of $Q(x)$, or its inverse function $x(Q)$, was given by Parisi [4] in relating these to a replica-free quantity. In terms of a multi-phase picture, a probability $P(q)$ can be defined for the spin overlap function between two phases to have the value q ($Q = \beta J q$). Then it was shown that $P(q) = \mathrm{d}x/\mathrm{d}q$. We see from (26) that $P(Q) = \mathrm{d}^2 A/\mathrm{d}Q^2$. If $A(Q)$ is regarded as a thermodynamic potential, this identifies $P(Q)$ as the susceptibility, and it is necessarily positive for stability, as a probability distribution is expected to be.

There is one form of $Q(x)$ that leads to greatest difficulties, which is unfortunately the one of greatest relevance thus far: namely the replica symmetric form $Q(x) = Q_0$, a constant. In that case,

$$F(x) = xQ_0 + (1-x)Q_0 = Q_0 \tag{27}$$

so $F(x)$ is independent of x. The dependence of x on Q is highly singular in that there is no solution for x when $Q \neq Q_0$, and all values of x are permitted for $Q = Q_0$. However, if $Q(x)$ is monotonic and has a non-zero derivative, the function $A(Q)$ can be computed straightforwardly. In particular, the limiting form of $A(Q)$ can be computed, as the function $Q(x)$ is allowed to tend towards a constant value, leading to a solution even for that singular limit.

Consider now the general form for the spin-glass free energy. From (11) and (25) we obtain

$$(-1)^k n^{-1} \sum \lambda_\alpha^k = A^k(Q_1) - kQ_0[A(Q_0)]^{k-1} - k\int_{Q_0}^{Q_1} \mathrm{d}Q\, A^{k-1}(Q) \tag{28}$$

where $Q_1 = F(1) = A(Q_1)$, and $\int_0^1 \mathrm{d}x Q(x) = F(0) = A(Q_0)$ (with Q_1 and Q_0 being the values of $Q(x)$ at $x = 1$ and 0 respectively). Integrating by parts, it is easily checked that

$$\int_0^1 \mathrm{d}x\, Q^2(x) = \int_{Q_0}^{Q_1} \mathrm{d}Q\, \frac{\mathrm{d}x}{\mathrm{d}Q} Q^2(x) = Q^2 \frac{dA}{\mathrm{d}Q}\bigg|_{Q_0}^{Q_1} - 2QA(Q)\bigg|_{Q_0}^{Q_1} + 2\int \mathrm{d}Q\, A(Q). \tag{29}$$

The full free energy, equation (12), can thus be cast into the form

$$H_{\text{eff}} = H_0(Q_0, A(Q_0)) + H_1(Q_1, A(Q_1)) + \int \mathrm{d}Q\, h(A(Q)). \tag{30}$$

The lowest-order replica symmetry breaking term was (from the previous sections) $-u \int \mathrm{d}x Q^4(x)$, which can be rewritten

$$-u \int \mathrm{d}Q\, \frac{\mathrm{d}^2 A}{\mathrm{d}Q^2} Q^4 = -4uQ^4 \frac{\mathrm{d}A}{\mathrm{d}Q}\bigg|_{Q_0}^{Q_1} + 4uQ^3 A(Q)\bigg|_{Q_0}^{Q_1} - 12u \int_{Q_0}^{Q_1} \mathrm{d}Q\, Q^2 A(Q). \tag{31}$$

Thus the perturbed free energy is of the form

$$H_{\text{eff}} = H_0(Q_0, A(Q_0)) + H_1(Q_1, A(Q_1)) + \int \mathrm{d}Q\, h(A(Q)) - 12u \int \mathrm{d}Q\, Q^2 A(Q). \tag{32}$$

Now we adopt the boundary condition (for zero external field) $Q_0 = 0$. In that case all contributions to H_0 vanish, and we are left with

$$H_{\text{eff}} = H_1 + \int_{Q_0}^{Q_1} \mathrm{d}Q\, [-12uQ^2 A(Q) + h(A(Q))] \tag{33}$$

and thus $A(Q)$ satisfies the equation (taking the functional derivative with respect to A)

$$-12uQ^2 + h' A(Q) = 0. \tag{34}$$

With

$$h(A(Q)) = \frac{A(Q)}{2} - \frac{1}{2} \frac{\beta J}{2z - \beta J A(Q)} \tag{35}$$

using (13) we have

$$-12uQ^2 + \frac{1}{2} - \frac{1}{2} \frac{(\beta J)^2}{(2z - \beta J A(Q))^2} = 0. \tag{36}$$

Solving the saddlepoint equation (34) for $A(Q)$, we obtain

$$A(Q) = \frac{2z}{\beta J} \pm \frac{1}{\sqrt{1 - 24uQ^2}} \tag{37}$$

from which the allowed solutions for $Q(x)$ are found to be either $Q(x) = 0$ (imposing the condition that the allowed function cannot be a decreasing function of x, on stability grounds [3]) or

$$Q(x) = \frac{1}{6\sqrt{2u}} \frac{x}{\sqrt{x^2 + 8u}}. \tag{38}$$

This is precisely the solution found in section 5.

7. Discussion

We have analysed the spherical model from the point of view of understanding what happens to the replica symmetric solution when one passes to the limit of Ising spins. The stability properties of the replica symmetric solution are investigated in the enlarged phase space of matrices parametrised *à la* Parisi. A more complete analysis would ideally dispense with this ansatz for replica symmetry breaking, however in the context of the replica method, we were unable to find a generalisation that would give reasonable results in the $n \to 0$ limit†. Within the limitations of the ansatz, however, the results obtained show that replica symmetry is broken when the spin lengths on each site are favoured to have a non-zero value. Requiring this effectively implies turning up the frustration in the system, as the most frustrated spins in the system will no longer set themselves automatically to zero. The mass of the symmetry breaking fluctuations depends on the spin length fluctuations, and is proportional to the fourth cumulant of the spin length distribution.

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Appendix

The calculation of the second variational derivative of L_{eff} is given here. Equation (15) gives this functional of $Q(x)$ up to quartic terms in Q, (without the added perturbation uQ^4 ,which can be easily included). Differentiating this with respect to $Q(x)$ and then $Q(y)$ leads to

$$\begin{aligned}
-\delta^2 L_{\text{eff}} &= \int \mathrm{d}x\,\mathrm{d}y \frac{\delta^2(-L_{\text{eff}})}{\delta Q(x)\delta Q(y)} \\
&= \int_0^1 \mathrm{d}x\,(\delta Q(x))^2 \left\{ \frac{1-(\beta J/2z)^2}{2} + \left(\frac{\beta J}{2z}\right)^3 \left(xQ(x) + \int_x^1 \mathrm{d}t\, Q(t) \right) \right. \\
&\quad \left. - \left(\frac{\beta J}{2z}\right)^4 \left[3xQ(x) \int_x^1 \mathrm{d}t\, Q(t) + \tfrac{3}{2} x^2 Q^2(x) + \frac{3}{2}\left(\int_x^1 \mathrm{d}t\, Q(t) \right)^2 \right] \right\} \\
&\quad + 2\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, \delta Q(x)\delta Q(y) \left[\left(\frac{\beta J}{2z}\right)^3 Q(y) - \left(\frac{\beta J}{2z}\right)^4 \right. \\
&\quad \left. \times \left(\frac{3}{2} y Q^2(y) + 3Q(y) \int_y^1 \mathrm{d}t\, Q(t) + \frac{3}{2} \int_0^y \mathrm{d}t\, Q^2(t) \right) \right].
\end{aligned} \tag{A1}$$

We wish to evaluate this quantity below the transition temperature, where $Q(x) \neq 0$. The form of $Q(x)$ is determined, of course, by the saddlepoint condition, obtained upon differentiating (15) with respect to $Q(x)$. One obtains thus

$$\frac{(\beta J/2z)^2 - 1}{2} + \left(\frac{\beta J}{2z}\right)^2 xQ(x) - \left(\frac{\beta J}{2z}\right)^3 \left[3xQ(x) \int_x^1 Q + \frac{3}{2}\left(\int_x^1 Q \right)^2 \right] = 0. \tag{A2}$$

† However, a recent analysis that is not based on replicas has yielded a non-trivial Parisi-like $Q(x)$ in the mean spherical model [13].

The saddlepoint condition requires, in fact, that the 'diagonal' term in the equation (A1) above vanish. The second term, when evaluated for the special case that $Q(x) = Q$, reduces to

$$\delta^2(-L) = \left(\frac{\beta J}{2z}\right)^2 \left[2\left(\frac{\beta J Q}{2z}\right) - 6\left(\frac{\beta J Q}{2z}\right)^2\right] \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\ \delta Q(x)\delta Q(y). \tag{A3}$$

Substituting the saddlepoint value for Q (as obtained for the truncated action close to the transition) gives the result in the text, equation (18).

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